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ON A CONVEXITY PRESERVING INTEGRAL OPERATOR

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Abstract

Let c be a complex number, with $Re\, c > 0$ and let g be an analytic function in the unit disc, $U = \{z \in \mathbb{C};\ |z| < 1\}$ with $g(0) = 0$, $g'(0) \neq 0$ and $g(z) \neq 0$, for $0 < |z| < 1$. In this paper we determine conditions an analytic function g needs to satisfy in order that the function F given by (1) be convex.

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1. Introduction and preliminaries

Let U be the unit disc of the complex plane:

$$U = \{z \in \mathbb{C}:\ |z| < 1\}.$$

Let $\mathcal{H}(U)$ denote the class of analytic functions in U. Also, let

$$A_n = \{f \in \mathcal{H}(U);\ f(z) = z + a_{n+1}z^{n+1} + \dots,\ z \in U\}$$

with $A_1 = A$,

$$K = \left\{f \in A,\ Re\, \frac{zf''(z)}{f'(z)} + 1 > 0,\ z \in U\right\}$$

denote the class of normalized convex functions in U,

$$C=\left\{f\in A;\ \exists\,\varphi\in K,\ Re\,\frac{f'(z)}{\varphi'(z)}>0,\ z\in U\right\}$$

denote the class of close-to-convex functions.

In order to prove our original result, we use the following lemma:

LEMMA A. ([9]) *If P is an analytic function in U, with $ReP(0)>0$ and if P satisfies*

$$Re\left[P(z)+\frac{zP'(z)}{P(z)}\right]>0,\quad z\in U,$$

then $ReP(z)>0$, $z\in U$.

Let c be a complex number, with $Re\,c>0$ and $g\in\mathcal{H}(U)$, with $g(0)=0$, $g'(0)\neq 0$ and $g(z)\neq 0$, for $0<|z|<1$. Consider the integral operator $I:\mathcal{H}(U)\to\mathcal{H}(U)$ defined by $F=I(f)$, where

$$F(z)=\frac{1}{[g(z)]^c}\int_0^z f(w)g(w)^{c-1}g'(w)dw,\quad z\in U,\ f\in\mathcal{H}(U).\tag{1}$$

It is well-known that in the particular case $g(z)=z$ and $c=1$, Libera [3] proved that the operator I preserves the starlikeness, the convexity and the close-to-convexity. This remarkable result was extended by many other authors (see, for example [1], [2], [4], [5], [6], [7], [12]-[14]).

For c a complex number, with $Re\,c>0$, and $g(z)=z$ similar results were obtained in [10] and [11] for the Bernardi integral operator.

In the case $c=1$, sufficient conditions on the function g such that I is a convexity-preserving operator were given in [8].

In [9] the author shows that if g satisfies the condition

$$Re\,[czg'(z)/g(z)]>0$$

in U and if the integral operator I preserves the convexity, then I also preserves the close-to-convexity.

In this paper we show that if g satisfies the conditions

$$Re\,\frac{czg'(z)}{g(z)}>0$$

and

$$Re\,\left[\frac{zg''(z)}{g'(z)}+1\right]>Re(c+1)\frac{zg'(z)}{g(z)}$$

in U and if the integral operator I preserves the close-to-convexity, then I also preserves the convexity.

2. Main result

THEOREM 1. *Let I be the integral operator defined by (1) and suppose that*

(i) $Re\,\frac{czg'(z)}{g(z)} > 0, \quad z \in U,\ Re\,c > 0,$

(ii) $Re\left[\frac{zg''(z)}{g'(z)} + 1\right] > Re\,\frac{(c+1)zg'(z)}{g(z)}, \quad z \in U,$

(iii) $I(C) \subset C$

then

$$I(K) \subset K.$$

P r o o f. If we let

$$G(z) = \frac{g(z)}{zg'(z)}, \quad z \in U,$$

then the condition (i) implies $G \in \mathcal{H}(U)$ and $G(z) \neq 0$ in U.

From (1), we obtain

$$zF'(z)G(z) + cF(z) = f(z), \quad z \in U$$

and

$$zF''(z)G(z) + [zG'(z) + G(z) + c]F'(z) = f'(z), \quad z \in U.$$

Let $f \in C$. Then there exists $\varphi \in K$, such that

$$Re\,\frac{f'(z)}{\varphi'(z)} > 0, \quad z \in U.$$

If we denote $\phi = I(\varphi)$, then

$$\phi(z) = \frac{1}{[g(z)]^c}\int_0^z \varphi(w)[g(w)]^{c-1}g'(w)dw, \quad Re\,c > 0. \tag{2}$$

Next we prove that $\phi \in K$.

Differentiating (2), we obtain

$$z\phi''(z)G(z) + [zG'(z) + G(z) + c]\phi'(z) = \varphi'(z), \quad z \in U$$

which is equivalent to

$$G(z)\phi'(z)\left[\frac{z\phi''(z)}{\phi'(z)} + \frac{zG'(z)}{G(z)} + 1 + \frac{c}{G(z)}\right] = \varphi'(z). \tag{3}$$

If we let

$$P(z) = \frac{z\phi''(z)}{\phi'(z)} + \frac{zG'(z)}{G(z)} + 1 + \frac{c}{G(z)}, \quad z \in U, \tag{4}$$

then (3) becomes

$$G(z)\cdot\phi'(z)\cdot P(z)=\varphi'(z),\quad z\in U. \tag{5}$$

Differentiating (5), we obtain

$$\frac{zG'(z)}{G(z)}+\frac{z\phi''(z)}{\phi'(z)}+\frac{zP'(z)}{P(z)}=\frac{z\varphi''(z)}{\varphi'(z)},\quad z\in U$$

which is equivalent to

$$\frac{zG'(z)}{G(z)}+\frac{c}{G(z)}+\frac{z\phi''(z)}{\phi'(z)}+1+\frac{zP'(z)}{P(z)}=\frac{z\varphi''(z)}{\varphi'(z)}+1+\frac{c}{G(z)},\quad z\in U. \tag{6}$$

Using (4) in (6), we obtain

$$P(z)+\frac{zP'(z)}{P(z)}=\frac{z\varphi''(z)}{\varphi'(z)}+1+\frac{c}{G(z)},\quad z\in U. \tag{7}$$

Using condition (i) from hypothesis and since φ is convex, we have

$$\Re\left[P(z)+\frac{zP'(z)}{P(z)}\right]=Re\left[\frac{z\varphi''(z)}{\varphi'(z)}+1+\frac{czg'(z)}{g(z)}\right]>0,\quad z\in U,$$

i.e.

$$Re\left[P(z)+\frac{zP'(z)}{P(z)}\right]>0,\quad z\in U. \tag{8}$$

Letting $z=0$ in (8), we deduce

$$Re\,P(0)>0,\quad z\in U.$$

We have now the conditions from the hypothesis of Lemma A and applying it we obtain

$$ReP(z)>0,\quad z\in U.$$

From $G(z)=\frac{g(z)}{zg'(z)}$, we have

$$\frac{zG'(z)}{G(z)}=\frac{zg'(z)}{g(z)}-\frac{zg''(z)}{g'(z)}-1,\quad z\in U.$$

Using (4) and the condition $ReP(z)>0$, $z\in U$ we obtain

$$Re\left[\frac{z\phi''(z)}{\phi'(z)}+1+\frac{zG'(z)}{G(z)}+\frac{c}{G(z)}\right]>0,$$

and using (ii), we obtain

$$Re\left[\frac{z\phi''(z)}{\phi'(z)}+1\right] > Re\left[\frac{zg''(z)}{g'(z)}+1-\frac{(c+1)zg'(z)}{g(z)}\right] > 0, \quad z\in U,$$

i.e.

$$Re\left[\frac{z\phi''(z)}{\phi'(z)}+1\right] > 0, \quad z\in U$$

which shows that $\phi \in K$. ∎

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